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ON A DIVISION PROPERTY OF CONSECUTIVE INTEGERS

BY

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ABSTRACT

Pillai and Brauer proved that for $m \geqq 17$ we can find blocks B_m of m consecutive integers such that no element in the block is pairwise prime with each of the other elements. The following basic generalization is proved: For each $d > 1$ there is a number $G(d)$ such that for every $m \geqq G(d)$ there exist infinitely many blocks B_m of m consecutive integers, such that for each $r \in B_m$ there exists $s \in B_m$, $(r, s) \geqq d$.

Introduction

In the year 1940 the Indian Mathematician S. Pillai [3] formulated the following question: Let B_m be an arbitrary block of m positive consecutive integers; can we find an integer $r \in B_m$ such that for all $s \in B_m$, $(s, r) = 1$.

Pillai proved that this is true for $2 \leqq m \leqq 16$, but he proved also that for $17 \leqq m \leqq 429$ there exist infinitely many blocks B_m in which for any $r \in B_m$ there exists $s \in B_m$ such that $(r, s) \geqq 2$ $(s \neq r)$.

In the year 1941 A. Brauer [1] proved that for any $m \geqq 17$ there exist infinitely many blocks B_m in which for any $r \in B_m$ there exists $s \in B_m$ such that $(r, s) \geqq 2$ $(s \neq r)$.

In the year 1969 R. J. Evans [2] gave a simpler proof for $m \geqq 17$. Here I prove the stronger result which is:

Let $d \geqq 1$ be any integer. There exists a number $G(d)$ such that for $m \geqq G(d)$ infinitely many blocks B_m exist in which for any $r \in B_m$ there exists $s \in B_m$ such that $(r, s) \geqq d$.

I prove also a theorem of this kind concerning more general series.

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LEMMA. *Let* $d \geqq 1$ *be a given integer. A number* $N(d)$ *exists such that for* $n/2 \geqq N(d)$ *there are at least* $4d-5$ *primes between* $n/2$ *and* $3n/4$.

PROOF. This follows immediately from the prime number theorem by which we can show that $\lim_{n\to\infty}(\Pi(3n/4)-\Pi(n/2))=\infty$.

THEOREM 1. *Let* $d>1$ *be a given integer. A number* $G(d)$ *exists such that for* $m \geqq G(d)$ *infinitely many blocks* B_m *exist such that for* $r \in B_m$ *there exists* $s \in B_m$ *such that* $(r,s) \geqq d$.

PROOF. Denote by $p_d, \cdots, p_t$ the prime numbers which are not smaller than d and smaller than $n/2$ while $n/2 \geqq N(d)$, that is, $d \leqq p < n/2$ for this prime.

Denote by $p_1, \cdots, p_k$ the prime numbers smaller than d which satisfy $(p_i, d-1)=1$, $i=1,\cdots,k$.

Denote by e_i the smallest integer such that $p_i^{e_i} > d-1$, $i=1,\cdots,k$. Let $R=(d-1)^2\Pi_{i=1}^{k} p_i^{e_i}$.

Denote by $q_1, \cdots, q_{4d-5}$ the first primes which satisfy $n/2 \leqq q_j \leqq 3n/4$, $j=1,\cdots,4d-5$. Now consider the $4d-4$ congruences:

(1) $x \equiv -(d-1) \pmod{q_1}$,

(2) $x \equiv -(d-2) \pmod{q_2}$,

. . . .

$(d-1)$ $x \equiv -1 \pmod{q_{d-1}}$,

(d) $x \equiv 1 \pmod{q_d}$,

. . . .

$(2d-2)$ $x \equiv (d-1) \pmod{q_{2d-2}}$,

$(2d-1)$ $x \equiv -q_1 \pmod{q_{2d-1}}$,

. . . .

$(4d-5)$ $x \equiv -q_{2d-3} \pmod{q_{4d-5}}$,

$(4d-4)$ $x \equiv 0 \pmod{R\, p_d \cdots p_t}$.

Since the congruences are modulo pairwise prime integer, then by the Chinese Remainder Theorem infinitely many solutions exist.

Consider the following block B_m of $M(d)$ consecutive integers:

$$\{x-[n/4], \cdots, x-d, x-(d-1), \cdots, x-1, x, x+1, \cdots, x+q_1, \cdots$$
$$\cdots, x+q_2, \cdots, x+q_{2d-2}-1\}.$$

We notice that $M(d) > 3n/4$.

It will be shown that for any $r \in B_m$ there is $s \in B_m$ such that $(r,s) \geqq d$. If $r=x$ we choose $s=x+d$; clearly $d \mid x$ and $d \mid x+d$ therefore $(r,d) \geqq d$.

For $1 \leqq j \leqq d-1$ we consider the following cases:

(1) If $r = x + j$ we choose $s = x + q_{d-j} + j$, $s \in B_m$, $(s, r) = q_{d-j} > d$.

(2) If $r = x - j$ we choose $s = x + q_{d+j-1} - j$, $s \in B_m$, $(s, r) = q_{d-j+1} > d$. Since $q_{d-j} + j < q_{2d-1}$, $q_{d+j-1} - j < q_{2d-1}$ it follows that both $s \in B_m$ in both cases.

(3) If $r = x + q_j$, $j = 1, \cdots, 2d-3$ we choose $s = x - (q_{2d+j-2} - q_j)$. Since $[n/4] > q_{2d+j-2} - q_j$ it follows that $s \in B_m$ and $(r, s) = q_{2d+j-2} > d$.

(4) If $r = x \pm j$ when j is not of the cases described above, then we choose $s = x$ and $(r, s) \geqq d$.

Therefore for each $r \in B_m$ there exists $s \in B_m$ such that $(r, s) \geqq d$. We also notice that we can make the left side of B_m far smaller. We can replace $x - [n/4]$ by $x - q_1 + 1$ without changing the truth of the theorem; that is, we shall get $n/4$ numbers greater than $M(d)$ for which the theorem holds. Now for $n_1 > n$, for which $q_2 \geqq n_1/2 > q_1$, we can take the block

$$\{x - [n_1/4], \cdots, x - d, \cdots, x - 1, x, x + 1, \cdots, x + q_{2d-1} - 1\}$$

which is constructed in the same manner as the first block. The length of this block is $q_{2d-1} + [n_1/4] < q_{2d-2} + q_1 - 1$, which was the length of the largest block we had. We can make the left side of this block smaller by replacing $x - [n_1/4]$ with $x - q_2 + 1$. The length of the largest block is $q_{2d-1} + q_2 - 1 > q_{2d-2} + q_1 - 1$. Repeating this argument we get the truth of the theorem for $m \geqq M(d)$.

Definition. Let $\{A_n\}_{n=1}^{\infty}$ be a non-decreasing series of positive integers. We say that A_n is a perfect series if for any positive integers n, k, $A_n \mid A_{kn}$.

Theorem 2. *Let $\{A_n\}_{n=1}^{\infty}$ be a perfect series such that $\lim_{n\to\infty} A_n = \infty$. Then for any given integer $d > 1$ there exists a number $k(d)$ such that for $m \geqq k(d)$ there are infinitely many blocks B_m of m consecutive terms of the series such that for each $A_r \in B_m$ there exists $A_s \in B_m$ and $(A_r, A_s) \geqq d$.*

Proof. From Theorem 1 we know that we can find blocks B_m of m consecutive integers such that for each $r \in B_m$ there is $s \in B_m$ and $(r, s) \geqq b$. Suppose that the block B_m is $n, n+1, \cdots, n+m-1$; we consider the term A_n, $A_{n+1}, \cdots, A_{n+m-1}$. For each $A_r \in B_m$ there is $A_s \in B_m$ such that $(A_r, A_s) \geqq A_b$. Since $A_n \to \infty$ then for a sufficiently large b, $A_b \geqq d$, hence $(A_r, A_s) \geqq A_b \geqq d$.

Corollary. *The Fibonacci series is a perfect series, since $F_1 = 1$, $F_2 = 1$,*

$F_3 = 2$. *It follows from Theorem 2 that for* $m \geqq G(3)$ *there exists a block* B_m *of* m *consecutive Fibonacci numbers such that for each* $F_r \in B_m$ *there is* $F_s \in B_m$, $(F_r, F_s) \geqq F_3 = 2$.

We notice that $A_n = 2^n - 1$, $A_n = (10^n - 1)/9$ are also perfect series.

Upper bounds for $G(d)$ and $g(d)$

It follows from the proof of Theorem 1 that we can change the restriction $\Pi(3n/4) - \Pi(n/2) \geqq 4d - 5$ to $\Pi(n) - \Pi(n/2) \geqq 4d - 5$, where now the primes $q_1, \cdots, q_{4d-5}$ are the first to satisfy $n/2 \leqq q_i \leqq n$, $i = 1, \cdots, 4d - 5$.

For any n for which $\Pi(n) - \Pi(n/2) \geqq 4d - 5$ the arguments of the proof can be adopted, but we cannot enlarge the length of the blocks. Therefore it might be that for some $m < G(d)$ there are blocks B_m with the required property.

Let $g(d)$ denote the smallest number for which there is a block $B_{g(d)}$ such that for each $r \in B_{g(d)}$, $\exists s \in B_{g(d)}$ and $(r, s) \geqq d$.

THEOREM 3.

$$d \geqq 2, \qquad g(d) < 45d \lg d,$$

$$d \geqq 2, \qquad G(d) < 54d \lg d.$$

PROOF. We use two inequalities of Rosser–Schoenfeld [4]:

(1) For $x \geqq 21$, $\Pi(2x) - \Pi(x) > 3x/(5 \lg x)$;

(2) For $x \geqq 67$, $x/(\lg x - 0.5) < \Pi(x) < x/(\lg x - 1.5)$.

For $g(d)$ it is sufficient to consider the inequality $\Pi(n) - \Pi(n/2) \geqq 4d - 5$; put $n = 2x$, $n/2 = x$: we get

$$\Pi(2x) - \Pi(x) > 3x/(5 \lg x) \geqq 4d - 5;$$

put $x = 15d \cdot \lg d$ we find

$$\frac{9(d \lg d)}{\lg(15d \cdot \lg d)} \geqq 4d - 5$$

which is true for $d \geqq 2$. Hence $n = 2x = 30d \cdot \lg d$, $g(d) \leqq 3n/2 = 45d \lg d$.

For $G(d)$ we consider $\Pi(3n/4) - \Pi(n/2) \geqq 4d - 5$. It is clear that for $n/2 \geqq 67$,

$$\Pi(3n/4) - \Pi(n/2) \geqq \frac{3n}{4(\lg(3n/4) - 0.5)} - \frac{n}{2(\lg(n/2) - 1.5))}$$

$$= \frac{3n}{4(\lg n + \lg(3/4) - 0.5)} - \frac{2n}{4(\lg n - \lg 2 - 1.5)}$$

$$> \frac{3n}{4 \cdot \lg n} - \frac{2n}{4(\lg n - 2.2)}$$

$$> \frac{n}{4.51 \lg n}.$$

Therefore we consider

$$\frac{n}{4.5 \lg n} \geqq 4d - 5.$$

For $n = 54d \lg d$ we get

$$\frac{12d \lg d}{\lg(54d \lg d)} \geqq 4d - 5,$$

which is true for $d \geqq 3$.

We easily verify that this holds for $d = 2$. Consequently $G(d) \leqq n = 54d \lg d$. Indeed the upper bound for $g(d)$ and $G(d)$ can be reduced further, as one can see from the following statement:

COROLLARY.
$g(3) \leqq 81$ *since* $27 < 29$, 31, 37, 41, 43, 47, $53 < 54$,
$g(4) \leqq 153$ *since* $51 < 53$, 59, 61, 67, 71, 73, 79, 83, 89, 97, $101 < 102$,
$g(5) \leqq 228$ *since there are* 15 *primes between* 76 *and* 152,
$g(6) \leqq 288$ *since there are* 19 *primes between* 96 *and* 192.

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